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# 行列のHLAWKA不等式 (作用素論の 発展と諸問題)

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# 行列の HLAWEKA 不等式

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## 1. INTRODUCTION

For arbitrary complex numbers  $x, y, z$ , the inequality

$$|x + y| + |y + z| + |z + x| \leq |x| + |y| + |z| + |x + y + z|$$

is well known as Hlawka's inequality. Djoković [2] proved the following inequalities which contain the above one as a special case:

$$(1) \quad \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |x_{i_1} + \dots + x_{i_k}| \leq \binom{n-2}{k-2} \left( \frac{n-k}{k-1} \sum_{i=1}^n |x_i| + \left| \sum_{j=1}^n x_j \right| \right)$$

for complex numbers  $x_1, \dots, x_n$  and  $2 \leq k \leq n$ .

In this paper, we pay attention to the special case  $k = n - 1$  of (1), namely

$$(2) \quad \|\mathbf{a} - \text{Tr } \mathbf{a}\|_1 \leq \|\mathbf{a}\|_1 + (n-2) |\text{Tr } \mathbf{a}|$$

for a vector  $\mathbf{a} = (a_1, \dots, a_n)$  in  $\mathbb{C}^n$  ( $n \geq 2$ ), where  $\|\cdot\|_1$  is  $\ell_1$  norm on  $\mathbb{C}^n$  and  $\text{Tr } \mathbf{a} = \sum_{i=1}^n a_i$ . A weighted extension of the inequality (2) is known and is stated as follows:

**Proposition 1.** *Let  $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$  and  $x_1, x_2, \dots, x_n \in \mathbb{C}$  ( $n \geq 2$ ). Then*

$$(3) \quad \sum_{i=1}^n \alpha_i |x_i| - \sum_{j=1}^n \alpha_j x_j \leq \beta \sum_{i=1}^n \alpha_i |x_i| + \left( \sum_{i=1}^n \alpha_i - 2\alpha \right) \left| \sum_{j=1}^n \alpha_j x_j \right|,$$

where  $\alpha = \min\{\alpha_i : \alpha_i > 0\}$  and  $\beta = \max\{2\alpha - 1, 1\}$ .

In view of the inequality (2), it might be natural to write the matrix version of Hlawka's inequality as follows:

$$(4) \quad \|A - \text{Tr } A\|_1 \leq \|A\|_1 + (n-2) |\text{Tr } A|$$

for a complex  $n \times n$  matrix  $A$ , where  $\text{Tr } A$  is the trace of  $A$  and  $\|A\|_1$  is the trace norm of  $A$ , i.e.  $\|A\|_1 = \text{Tr } |A|$  with  $|A| = (A^* A)^{1/2}$ . In this paper we prove the inequality (4) in a more general form stated in (5) of the following theorem. Indeed, the inequality (5) is not only a matrix extension of (3) but also a weighted extension of (4).

**Theorem 2.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$  and let  $A, B$  be complex  $n \times n$  matrices. If  $AB = BA$  and  $B \geq 0$ , then*

$$(5) \quad \text{Tr}(B|A - \text{Tr } BA|) \leq \max\{2\gamma(B) - 1, 1\} \text{Tr } B|A| + (\text{Tr } B - 2\gamma(B)) |\text{Tr } BA|,$$

where  $\gamma(B)$  denotes the minimum positive eigenvalue of  $B$ .

## PROOF OF THEOREM 2

*Proof.* Since  $B \geq 0$  and  $AB = BA$ , we can write  $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$  with invertible  $B_1$  after some unitary conjugation. So it is enough to assume that  $B$  is invertible.

By approximation, we may assume that  $\text{Tr } B \neq 1$ . If we put  $A_\varepsilon = A + \varepsilon I$ , then  $A_\varepsilon - \text{Tr } BA_\varepsilon = A - \text{Tr } BA + \varepsilon(1 - \text{Tr } B)$  is invertible for small  $\varepsilon > 0$ . So it suffices to prove the case where  $A - \text{Tr } BA$  is invertible. Since  $AB = BA$ , there is a unique unitary matrix  $U$  such that  $U(A - \text{Tr } BA) = |A - \text{Tr } BA|$  and  $UB = BU$ . Hence there exists a unitary matrix  $V$  and diagonal matrices  $D_B$  and  $D_U$  such that

$$B = V^* D_B V, \quad U = V^* D_U V.$$

So we have

$$\begin{aligned} \text{Tr}(B|A - \text{Tr } BA|) &= \text{Tr}(BU(A - \text{Tr } BA)) \\ &= \text{Tr}(V^* D_B D_U V(A - \text{Tr } V^* D_B V A)) \\ &= \text{Tr}(D_B D_U (V A V^* - \text{Tr } D_B V A V^*)). \end{aligned}$$

In this way, we can suppose that  $B = \text{diag}(b_1, b_2, \dots, b_n)$  with  $b_i > 0$  and  $U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$ . Then, by using Proposition 1, we have

$$\begin{aligned} \text{Tr}(B|A - \text{Tr } BA|) &= \left| \sum_{i=1}^n e^{i\theta_i} b_i (A - \text{Tr } BA)_{ii} \right| \\ &\leq \sum_{i=1}^n b_i |a_{ii} - \sum_{j=1}^n b_j a_{jj}| \\ &\leq \max\{2 \min b_i - 1, 1\} \sum_{i=1}^n b_i |a_{ii}| + \left( \sum_{i=1}^n b_i - 2 \min b_i \right) \sum_{j=1}^n b_j |a_{jj}| \\ &\leq \max\{2\gamma(B) - 1, 1\} \sum_{i=1}^n b_i |a_{ii}| + (\text{Tr } B - 2\gamma(B)) |\text{Tr } BA|. \end{aligned}$$

Moreover, note that  $B$  commutes with  $A$  and  $A^*$  so that  $|BA| = B|A|$ . Since

$$\sum_{i=1}^n |\langle X e_i, e_i \rangle| \leq \text{Tr } |X|$$

for a general matrix  $X$  with the canonical basis  $e_1, \dots, e_n$  of  $\mathbb{C}^n$ , it follows that

$$\sum_{i=1}^n b_i |a_{ii}| = \sum_{i=1}^n |\langle B A e_i, e_i \rangle| \leq \text{Tr } |BA| = \text{Tr } B|A|.$$

Therefore, the desired inequality (5) is obtained.  $\square$

*Remark.* The inequality (5) fails to hold for some non-commuting pairs of matrices. For example, choose

$$A = \begin{pmatrix} 1 & 0.5 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

**Corollary 3.** For every  $n \in \mathbb{N}$  with  $n \geq 2$  and every complex  $n \times n$  matrix  $A$ ,

$$(6) \quad \|A - \operatorname{Tr} A\|_1 \leq \|A\|_1 + (n-2)|\operatorname{Tr} A|$$

and

$$(7) \quad \|A\|_1 \leq \|A - \frac{1}{n-1} \operatorname{Tr} A\|_1 + \frac{n-2}{n-1} |\operatorname{Tr} A|.$$

*Proof.* The inequality (6) is a specialization of (5) to the case  $B = I$ . Replace  $A$  by  $A - \frac{1}{n-1} \operatorname{Tr} A$  in (6) to obtain the inequality (7).  $\square$

Let  $A$  be a complex  $n \times n$  matrix with  $n \geq 2$ , and consider the function  $f(t) = \|A - t \operatorname{Tr} A\|_1$  for  $t \geq 0$ . When  $0 < t \leq 1$ , it follows from the convexity of  $f$  and (6) that

$$\frac{f(t) - f(0)}{t} \leq \frac{f(1) - f(0)}{1} \leq (n-2)|\operatorname{Tr} A|.$$

Therefore,

$$\|A - t \operatorname{Tr} A\|_1 \leq \|A\|_1 + t(n-2)|\operatorname{Tr} A| \quad (0 \leq t \leq 1).$$

This inequality is also obtained by putting  $B = tI$  in (5). Similarly, from the convexity of  $f$  and (7), we can show that

$$\|A\|_1 \leq \|A - t \operatorname{Tr} A\|_1 + t(n-2)|\operatorname{Tr} A| \quad (t \geq \frac{1}{n-1}).$$

### 3. RELATED INEQUALITIES

Let  $M_n(\mathbb{C})$  be the space of complex  $n \times n$  matrices. For  $1 \leq p < \infty$  let  $\|A\|_p$  denote the Schatten  $p$ -norm of  $A \in M_n(\mathbb{C})$ , i.e.  $\|A\|_p = (\operatorname{Tr} |A|^p)^{1/p}$ . Also, the operator norm of  $A$  is denoted by  $\|A\|_\infty$ .

In this section, we discuss some inequalities comparing  $\|A - \operatorname{Tr} A\|_p$  with  $\|A\|_p$ . To get such inequalities, we introduce some norms  $\|\cdot\|_{(1,1)}$  and  $|||\cdot|||$  on  $M_n(\mathbb{C})$  and determine their dual norms.

Define the norm  $\|\cdot\|_{(1,1)}$  on  $M_n(\mathbb{C})$  by

$$\|A\|_{(1,1)} = \|A - \operatorname{Tr} A\|_1 \quad (A \in M_n(\mathbb{C})).$$

Note that the linear mapping  $\Phi(A) = A - \operatorname{Tr} A$  on  $M_n(\mathbb{C})$  has the inverse  $\Phi^{-1}(A) = A - \frac{1}{n-1} \operatorname{Tr} A$ , so  $\|\cdot\|_{(1,1)}$  is actually a norm on  $M_n(\mathbb{C})$ . Consider the canonical duality  $\langle A, B \rangle = \operatorname{Tr} AB$  for  $A, B \in M_n(\mathbb{C})$ . Then we have

$$\begin{aligned} \max\{|\langle A, B \rangle| : \|A\|_{(1,1)} \leq 1\} &= \max\{|\langle A, B \rangle| : \|A - \operatorname{Tr} A\|_1 \leq 1\} \\ &= \max\{|\langle A - \frac{1}{n-1} \operatorname{Tr} A, B \rangle| : \|A\|_1 \leq 1\} \\ &= \max\{|\langle A, B - \frac{1}{n-1} \operatorname{Tr} B \rangle| : \|A\|_1 \leq 1\} \\ &= \|B - \frac{1}{n-1} \operatorname{Tr} B\|_\infty. \end{aligned}$$

This says that the dual norm of  $\|\cdot\|_{(1,1)}$  on  $M_n(\mathbb{C})$  is equal to

$$(8) \quad \|A\|_{(1,1)}^* = \|A - \frac{1}{n-1} \text{Tr } A\|_\infty.$$

We next define the norm  $|||\cdot|||$  on  $M_n(\mathbb{C})$  by

$$|||A||| = \|A\|_1 + (n-2)|\text{Tr } A| \quad (A \in M_n(\mathbb{C})).$$

Then the inequality (4) is rewritten as  $\|A\|_{(1,1)} \leq |||A|||$ , so the dual form of (4) is given as

$$(9) \quad |||A|||^* \leq \|A\|_{(1,1)}^*,$$

where  $|||\cdot|||^*$  is the dual norm of  $|||\cdot|||$  with respect to the canonical duality. Since  $\|A\|_{(1,1)}^* = \|A - \frac{1}{n-1} \text{Tr } A\|_\infty$  by (8), the above (9) is equivalent to

$$(10) \quad |||A - \text{Tr } A|||^* \leq \|A\|_\infty.$$

Now let us determine the dual norm  $|||\cdot|||^*$ . To do so, define a semi-norm on the direct sum  $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$  by

$$|||X \oplus Y|||_1 = \|X\|_1 + (n-2)|\text{Tr } Y| \quad (X, Y \in M_n(\mathbb{C})).$$

Clearly, the mapping  $X \mapsto X \oplus X$  is isometric from  $(M_n(\mathbb{C}), |||\cdot|||)$  into  $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), |||\cdot|||_1)$ . Let  $A \in M_n(\mathbb{C})$  satisfy  $|||A|||^* \leq 1$ . Then the norm of the functional  $X \oplus X \mapsto \langle X, A \rangle$  on the subspace  $\{X \oplus X : X \in M_n(\mathbb{C})\}$  of  $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), |||\cdot|||_1)$  is equal to  $|||A|||^*$ . By the Hahn-Banach extension theorem, this can be extended to a linear functional  $\varphi$  on  $(M_n(\mathbb{C}) \oplus M_n(\mathbb{C}), |||\cdot|||_1)$  which has the norm  $\leq 1$ , namely

$$(11) \quad |\varphi(X \oplus Y)| \leq |||X \oplus Y|||_1 = \|X\|_1 + (n-2)|\text{Tr } Y|.$$

On the other hand, it is obvious that we can choose matrices  $B, C \in M_n(\mathbb{C})$  such that  $\varphi(X \oplus Y) = \langle X, B \rangle + \langle Y, C \rangle$ . Then (11) implies that  $B$  and  $C$  must satisfy  $\|B\|_1^* = \|B\|_\infty \leq 1$  and  $|\langle Y, C \rangle| \leq (n-2)|\text{Tr } Y|$ . The latter implies  $C = \zeta I$  for some  $|\zeta| \leq n-2$ , so we have  $\langle X, A \rangle = \langle X, B \rangle + \langle X, \zeta I \rangle$  for every  $X \in M_n(\mathbb{C})$ . This means that  $A = B + \zeta I$  for some  $|\zeta| \leq n-2$ . Conversely, it is immediate to see that if  $A = B + \zeta I$  with  $\|B\|_\infty \leq 1$  and  $|\zeta| \leq n-2$ , then  $|||A|||^* \leq 1$ . In this way, we conclude the following proposition.

**Proposition 4.**

$$\{A \in M_n(\mathbb{C}) : |||A|||^* \leq 1\} = \{X + \zeta I : |\zeta| \leq n-2, X \in M_n(\mathbb{C}), \|X\|_\infty \leq 1\}.$$

Put  $t := |||A - \text{Tr } A|||^*$ . From the above proposition, we have

$$\left\| \frac{1}{t}(A - \text{Tr } A) - \zeta \right\|_\infty \leq 1,$$

for some  $\zeta$  with  $|\zeta| \leq n - 2$ . So we have

$$\min_{|\zeta| \leq t} \|A - \text{Tr } A - (n - 2)\zeta\|_\infty \leq \|A - \text{Tr } A\|^*$$

Since the inequality (4) can be rewritten as  $t \leq \|A\|_\infty$ , we have

$$\min_{|\zeta| \leq \|A\|_\infty} \|A - \text{Tr } A - (n - 2)\zeta\|_\infty \leq \|A - \text{Tr } A\|^*.$$

This and (6) imply that

$$\|A - \text{Tr } A\|_p \leq (n - 1)\|A\|_p \quad (p = 1, \infty)$$

for all  $A \in M_n(\mathbb{C})$ . So the following is a consequence of the complex interpolation method (cf. [5, Appendix to IX.4]).

**Proposition 5.** *For every complex  $n \times n$  matrix  $A$  with  $n \geq 2$ ,*

$$(12) \quad \|A - \text{Tr } A\|_p \leq (n - 1)\|A\|_p \quad (1 \leq p \leq \infty).$$

As in the last part of Sect. 1 we can get

$$(13) \quad \|A - t \text{Tr } A\|_p \leq (tn - 2t + 1)\|A\|_p \quad (1 \leq p \leq \infty, 0 \leq t \leq 1)$$

from (12) and the convexity of  $t \mapsto \|A - t \text{Tr } A\|_p$ . According to (8) the dual form of (13) is given as

$$\|A - \frac{t}{tn - 1} \text{Tr } A\|_q \geq \frac{1}{tn - 2t + 1} \|A\|_q$$

for  $1 \leq q \leq \infty$  and  $0 \leq t \leq 1$  with  $t \neq 1/n$ . Rewriting this we obtain

$$\|A - t \text{Tr } A\|_p \geq \frac{tn - 1}{2tn - 2t - 1} \|A\|_p \quad (1 \leq p \leq \infty, t \geq \frac{1}{n-1}).$$

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